Journal of Mathematical Sciences: Advances and Applications
Volume 4, Number 1, 2010, Pages 191-199

POSSIBILITY MEASURE, PRODUCT POSSIBILITY SPACE AND THE NOTION OF INDEPENDENCE

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Abstract

This paper discusses possibility measure, product possibility space, and the notion of independence in their full details.

1. Introduction

Fuzzy set was first introduced by Zadeh [26] in 1965. This notion has been very useful in human decision making under uncertainty. We can see lots of papers, which use this fuzzy set theory in Iwamura and Liu [3], [4], Liu and Iwamura [12], [13], [16], Ji et al. [9], Gao and Iwamura [2], Wang and Iwamura [24], Wen and Iwamura [25], and others. We also have some books on fuzzy decision making under fuzzy environments

2010 Mathematics Subject Classification: 28A35.

Keywords and phrases: uncertainty theory, possibility measure, product possibility space, fuzzy variable, fuzzy vector, independence.

Received October 12, 2009

such as Dubois and Prade [1], Zimmermann [28], Sakawa [23], Kacprzyk [10], Liu and Esogbue [14].

Recently, Liu has founded a frequentionist fuzzy theory with huge amount of applications in fuzzy mathematical programming. We see it in books such as Liu [15], [18], [19]. In his book [19] published in 2004, Liu [19] has succeeded in establishing an axiomatic foundation for uncertainty theory, where they proposed a notion of independent fuzzy variables.

In this paper, we discuss on possibility axioms, construct product possibility space, and show that we not only have finitely many independent fuzzy variables [5], but also we have finitely many independent fuzzy vectors in a wide sense.

The rest of the paper is organized as follows. The next section provides a brief review on the results of possibility measure axioms with definitions of fuzzy variables, fuzzy vectors, independence of the two. Section 3 presents how we can get finitely many independent fuzzy vectors under possibility measures.

2. Possibility Measure and Product Possibility Space

We start with the axiomatic definition of possibility measure given by Liu [19] in 2004. Let Θ be an arbitrary nonempty set, and let $\mathcal{P}(\Theta)$ be the power set of Θ.

The three axioms for possibility measure are listed as follows:

Axiom 1. $\mathrm{Pos}\{\Theta\} = 1$.

Axiom 2. $\mathrm{Pos}\{\emptyset\} = 0$.

Axiom 3. $\mathrm{Pos}\{\bigcup_i A_i\} = \sup_i \mathrm{Pos}\{A_i\}$, for any collection $\{A_i\}$ in $\mathcal{P}(\Theta)$.

We call Pos a possibility measure over $\mathcal{P}(\Theta)$, if it satisfies these three axioms. We call the triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ a possibility space.

Theorem 2.1 (Liu [19]). *Let* Θ_i *be nonempty sets on which* $\mathrm{Pos}_i\{\cdot\}$ *satisfy the three axioms*, $i = 1, 2, \cdots, n$, *respectively*, *and let* $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. *Define* $\mathrm{Pos}\{\cdot\}$ *by*

$$\mathrm{Pos}\{A\} = \sup_{(\theta_1, \theta_2, \cdots, \theta_n) \in A} \mathrm{Pos}_1\{\theta_1\} \wedge \mathrm{Pos}_2\{\theta_2\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\}, \tag{2.1}$$

for each $A \in \mathcal{P}(\Theta)$, *where we use* $a \wedge b$ *in place of* $\min\{a, b\}$. *Then*, $\mathrm{Pos}\{\cdot\}$ *satisfies the three axioms for possibility measure*.

Therefore, a newly defined triplet $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ *is a possibility space*. $\mathrm{Pos}\{\cdot\}$ *is a possibility measure on* $\mathcal{P}(\Theta)$. *We call it product possibility measure derived from* $(\Theta_i, \mathrm{Pos}_i\{\cdot\}, \mathcal{P}(\Theta_i))$, $i = 1, \cdots, n$. *We call* $(\Theta, \mathcal{P}(\Theta), \mathrm{Pos})$ *the product possibility space derived from* $(\Theta_i, \mathrm{Pos}_i\{\cdot\}, \mathcal{P}(\Theta_i))$, $i = 1, \cdots, n$.

We show some lemmas to give detailed proof lines for this Theorem 2.1.

Lemma 2.1. *Let* $0 \leq a_i \leq 1$ *and let* $\epsilon > 0$, *for* $1 \leq i \leq n$. *Then*, *we get*

$$(\epsilon + a_1) \wedge (\epsilon + a_2) \wedge \cdots \wedge (\epsilon + a_n) \leq \epsilon + a_1 \wedge a_2 \wedge \cdots \wedge a_n. \tag{2.2}$$

Lemma 2.2. *Let* $A_i \in \mathcal{P}(\Theta_i)$ *for* $1 \leq i \leq n$. *Then*, *we get* $A_1 \times \cdots \times A_n \in \mathcal{P}(\Theta)$, *and*

$$\mathrm{Pos}\{A_1 \times \cdots \times A_n\} = \mathrm{Pos}_1\{A_1\} \wedge \mathrm{Pos}_2\{A_2\} \wedge \cdots \wedge \mathrm{Pos}_n\{A_n\}. \tag{2.3}$$

Lemma 2.3. $\mathrm{Pos}\{\cdot\}$ *on* $\mathcal{P}(\Theta)$ *at* (2.1) *is monotone with respect to set inclusion, i.e., for any sets* $A, B \in \mathcal{P}(\Theta)$ *with* $A \subset B$, *we get*

$$\mathrm{Pos}\{A\} \leq \mathrm{Pos}\{B\}. \tag{2.4}$$

Proof of Theorem 2.1.

$$\mathrm{Pos}\{\Theta\} = 1, \tag{2.5}$$

and

$$\mathrm{Pos}\{\emptyset\} = 0, \tag{2.6}$$

trivially hold.

Let $\{A_i\}$ be any collection of $\mathcal{P}(\Theta)$. By definition of $\mathrm{Pos}\{\cdot\}$, we get

$$\mathrm{Pos}\{\cup_i A_i\} = \sup_{(\theta_1, \cdots, \theta_n) \in \cup_i A_i} \mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\}, \tag{2.7}$$

and

$$\sup_i \mathrm{Pos}\{A_i\} = \sup_i \left(\sup_{(\theta_1, \cdots, \theta_n) \in A_i} \mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\} \right).$$

Let $\sup_i \left(\sup_{(\theta_1, \cdots, \theta_n) \in A_i} \mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\} \right) = b$. Then, we get

$$\sup_{(\theta_1, \cdots, \theta_n) \in A_i} \mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\} \le b \text{ for any } i, \tag{2.8}$$

and for any $\epsilon > 0$, there exists $i(\epsilon)$ such that

$$b - \epsilon < \sup_{(\theta_1, \cdots, \theta_n) \in A_{i(\epsilon)}} \left(\mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\} \right) \le b. \tag{2.9}$$

From (2.9), for this arbitrarily positive $\epsilon > 0$, we see that there exists $\theta_\epsilon^d = (\theta_{1,\epsilon}^d, \cdots, \theta_{n,\epsilon}^d) \in A_{i(\epsilon)} (\subset \cup_k A_k)$ such that

$$\sup_{(\theta_1, \cdots, \theta_n) \in A_{i(\epsilon)}} \left(\mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\} \right) - \epsilon < \mathrm{Pos}_1\{\theta_{1,\epsilon}^d\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_{n,\epsilon}^d\} \tag{2.10}$$

holds. So, (2.10) with (2.9) leads to

$$b - 2\epsilon < \sup_{(\theta_1, \cdots, \theta_n) \in A_{i(\epsilon)}} \left(\mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\} \right) - \epsilon < \mathrm{Pos}_1\{\theta_{1,\epsilon}^d\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_{n,\epsilon}^d\}. \tag{2.11}$$

On the other hand, for any $(\theta_1^d, \cdots, \theta_n^d) \in \cup_k A_k$, there exists i_0^d such that $(\theta_1^d, \cdots, \theta_n^d) \in A_{i_0^d}$. Using (2.8), we get

$$\mathrm{Pos}_1\{\theta_1^d\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n^d\} \leq \sup_{(\tilde{\theta}_1, \cdots, \tilde{\theta}_n) \in A_{i_0^d}} \mathrm{Pos}_1\{\tilde{\theta}_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\tilde{\theta}_n\},$$

i.e.,

$$\mathrm{Pos}_1\{\theta_1^d\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n^d\} \leq b. \tag{2.12}$$

Therefore, by (2.11) and (2.12), we get $b = \sup_{(\theta_1^d, \cdots, \theta_n^d) \in \bigcup_k A_k} \mathrm{Pos}_1\{\theta_1^d\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n^d\}$, which tells us that

$$\sup_i \left(\sup_{(\theta_1, \cdots, \theta_n) \in A_i} \mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\} \right) = \sup_{(\theta_1, \cdots, \theta_n) \in \bigcup_i A_i} \mathrm{Pos}_1\{\theta_1\} \wedge \cdots \wedge \mathrm{Pos}_n\{\theta_n\},$$

holds.

Let Θ_i be an arbitrary non empty set for $i = 1, \cdots, n$, Θ be the Cartesian product of $\Theta_1, \cdots, \Theta_n$ and $\widetilde{\mathrm{Pos}}\{\cdot\}$ be any possibility measure on $\mathcal{P}(\Theta)$. Define a set function $P_i\{\cdot\}$ on $\mathcal{P}(\Theta_i)$ by

$$P_i\{B\} = \widetilde{\mathrm{Pos}}\{\Theta_1 \times \cdots \times \Theta_{i-1} \times B \times \Theta_{i+1} \times \cdots \times \Theta_n\}, \tag{2.13}$$

$B \in \mathcal{P}(\Theta_i)$, $i = 1, \cdots, n$. Then, we get

Theorem 2.1. *$P_i\{\cdot\}$ on $\mathcal{P}(\Theta_i)$ satisfies Axioms* 1, 2, *and* 3. *Therefore, $P_i\{\cdot\}$ is a possibility measure on $\mathcal{P}(\Theta_i)$ for $i, \cdots, n$.*

Proof. $P_i\{\Theta_i\} = 1$ and $P_i\{\emptyset\} = 0$ are trivial. For any collection $\{B_j\}$ in $\mathcal{P}(\Theta_i)$, $\{\Theta_1 \times \cdots \times \Theta_{i-1} \times \bigcup_j B_j \times \Theta_{i+1} \times \cdots \times \Theta_n\}$ is a collection in $\mathcal{P}(\Theta)$. So, we get

$$\begin{aligned} P_i\{\textstyle\bigcup_j B_j\} &= \widetilde{\mathrm{Pos}}\{\Theta_1 \times \cdots \times \Theta_{i-1} \times \textstyle\bigcup_j B_j \times \Theta_{i+1} \times \cdots \times \Theta_n\} \\ &= \widetilde{\mathrm{Pos}}\left\{\textstyle\bigcup_j \{\Theta_1 \times \cdots \times \Theta_{i-1} \times B_j \times \Theta_{i+1} \times \cdots \times \Theta_n\}\right\} \end{aligned}$$

$$= \sup_j \widetilde{\text{Pos}}\{\Theta_1 \times \cdots \times \Theta_{i-1} \times B_j \times \Theta_{i+1} \times \cdots \times \Theta_n\}$$

$$= \sup_j P_i\{B_j\}.$$

Let $(\Theta_i, \mathcal{P}(\Theta_i), \text{Pos}_i\{\cdot\})$ be a possibility space, which satisfies Axioms 1, 2, and 3 for each i, $1 \le i \le n$. Let $\Theta = \Theta_1 \times \cdots \times \Theta_n$. Define a set function $\widetilde{\text{Pos}}\{\cdot\}$ by $\widetilde{\text{Pos}}\{A\} = \sup_{(\theta_1, \cdots, \theta_n) \in A} \text{Pos}_1\{\theta_1\} \wedge \text{Pos}_2\{\theta_2\} \wedge \cdots \wedge \text{Pos}_n\{\theta_n\}$, where $A \in \mathcal{P}(\Theta)$. Then, we get

Corollary 2.1. $P_i\{B\} = \text{Pos}_i\{B\}$, $B \in \mathcal{P}(\Theta_i)$ *holds for* $i = 1, 2, \ldots, n$.

A fuzzy variable is defined as a function from Θ of a possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ to the set of reals $\mathcal{R}$. An n-dimensional fuzzy vector is defined as a function from Θ of a possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$ to an n-dimensional Eucledian space $\mathcal{R}^n$. Let ξ be a fuzzy variable defined on the possibility space $(\Theta, \mathcal{P}(\Theta), \text{Pos})$.

Then its membership function is derived through the possibility measure Pos by

$$\mu(x) = \text{Pos}\{\theta \in \Theta \mid \xi(\theta) = x\},\ x \in \Re. \tag{2.14}$$

Theorem 2.2 (Liu [19]). *Let* $\mu : \Re \to [0, 1]$ *be a function with* sup $\mu(x) = 1$. *Then, there is a fuzzy variable whose membership function is* μ.

The fuzzy variables $\xi_1, \xi_2, \cdots, \xi_m$ are said to be independent, if and only if

$$\text{Pos}\{\xi_i \in B_i,\ i = 1, 2, \cdots, m\} = \min_{1 \le i \le m} \text{Pos}\{\xi_i \in B_i\},$$

for any subsets $B_1, B_2, \cdots, B_m$ of the set of reals $\mathcal{R}$. The fuzzy vectors $\boldsymbol{\xi}_i (1 \le i \le m)$ are said to be independent, if and only if

$$\text{Pos}\{\boldsymbol{\xi}_i \in B_i,\ i = 1, 2, \cdots, m\} = \min_{1 \le i \le m} \text{Pos}\{\boldsymbol{\xi}_i \in B_i\},$$

for any subsets B_i of $\mathcal{R}^{n_i} (1 \le i \le m)$. Here after, we use $a \wedge b$ in place of $\min\{a, b\}$.

Note 2.1. We have fuzzy variables ξ_1, ξ_2, which are not independent (Liu [19]). Let $\Theta = \{\theta_1, \theta_2\}$, $\text{Pos}\{\theta_1\} = 1$, $\text{Pos}\{\theta_2\} = 0.8$, and define ξ_1, ξ_2 by

$$\xi_1(\theta) = \begin{cases} 0, & \text{if } \theta = \theta_1, \\ 1, & \text{if } \theta = \theta_2, \end{cases} \quad \xi_2(\theta) = \begin{cases} 1, & \text{if } \theta = \theta_1, \\ 0, & \text{if } \theta = \theta_2. \end{cases}$$

Then, we have $\text{Pos}\{\xi_1 = 1, \xi_2 = 1\} = \text{Pos}\{\emptyset\} = 0 \neq 0.8 \wedge 1 = \text{Pos}\{\xi_1 = 1\} \wedge \text{Pos}\{\xi_2 = 1\}$.

3. Finitely Many Independent Fuzzy Vectors

Let $\boldsymbol{\xi}_i$ be a fuzzy vector from a possibility space $(\Theta_i, \mathcal{P}(\Theta_i), \text{Pos}_i)$ to the l_i-th dimensional Eucledian space $\mathcal{R}^{l_i}$, for $i = 1, 2, \cdots, n$. Define Θ by

$$\Theta = \Theta_1 \times \Theta_2 \times \cdots \Theta_n, \tag{3.1}$$

and $\tilde{\boldsymbol{\xi}}_i$ on Θ by

$$\tilde{\boldsymbol{\xi}}_i(\theta) = \xi_i(\theta_i), \quad \text{for any} \quad \theta = (\theta_1, \theta_2, \cdots, \theta_n) \in \Theta, 1 \leq i \leq n. \tag{3.2}$$

For any subset B_i of $\mathcal{R}^{l_i}$, we get [8].

Theorem 3.1. *The fuzzy vectors* $\tilde{\boldsymbol{\xi}}_i (1 \leq i \leq n)$ *given above are independent fuzzy vectors, i.e.*,

$$\text{Pos}\{\tilde{\boldsymbol{\xi}}_i(\theta) \in B_i (1 \leq i \leq n)\} = \text{Pos}\{\tilde{\boldsymbol{\xi}}_1 \in B_1\} \wedge \text{Pos}\{\tilde{\boldsymbol{\xi}}_2 \in B_2\} \wedge \cdots \wedge \text{Pos}\{\tilde{\boldsymbol{\xi}}_n \in B_n\}. \tag{3.3}$$

4. Conclusion

We have shown that Axiom 4 in Liu [19] can be proved through Axioms 1, 2, and 3. We have proved the fact that there exists finitely many independent fuzzy vectors. Through these proofs, we have shown that for possibility measures existence of finitely many independent fuzzy vectors depends on product possibility space. Although, the notion of independence was discovered by Zadeh and others [27] under the term of "noninteractiveness" or "unrelatedness", our notion of independence through product possibility space have brought about a grand world of fuzzy process, hybrid process, and uncertain process of Liu [21].

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